The Winding Number

Definition and Integer-Valuedness

Lecture Notes

November 5, 2026

1 Definition

Consider the Cauchy integral of $\varphi(z) \equiv 1$.

Lemma

Let γ be a (piecewise smooth) closed curve. Then for all $z_0 \notin \gamma$,

$$\frac{1}{2\pi i} \oint_\gamma \frac{dz}{z - z_0} \in \mathbb{Z}.$$

Notation

$n(\gamma, z_0)$ – the **index** of the point z_0 with respect to the curve γ:

$$n(\gamma, z_0) := \frac{1}{2\pi i} \oint_\gamma \frac{dz}{z - z_0}, \qquad (z_0 \notin \gamma).$$

For closed curves, this is also called the **winding number** of γ around z_0.

If γ is closed, by the Lemma, $n(\gamma, z_0) \in \mathbb{Z}$ for $z_0 \notin \gamma$.

Non-Proof of the Lemma (Heuristic)

$\big(\log(z-a)\big)' = \dfrac{1}{z-a}$. If $z : [a, b] \to \gamma$ is a parametrization, then

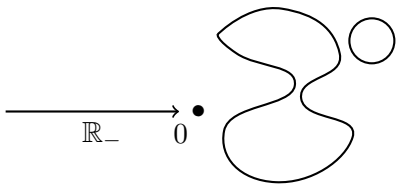
$$\oint_\gamma \frac{dz}{z - z_0} = \log\big(z(b) - z_0\big) - \log\big(z(a) - z_0\big) \in 2\pi i\,\mathbb{Z} \qquad (z(b) = z(a)).$$

This can be made rigorous using coverings.

Bonus (rigorous!) If $\gamma \subset \Omega$ and $\exists\, \ell : \Omega \to \mathbb{C}$ with $e^{\ell(z)} = z - z_0$ (a genuine, single-valued branch of $\log(z - z_0)$ defined on all of Ω), then

$$\oint_\gamma \frac{dz}{z - z_0} = \ell\big(z(b) - z_0\big) - \ell\big(z(a) - z_0\big) = 0.$$

Example. $\gamma \subset \mathbb{C} \setminus \mathbb{R}_- \implies n(\gamma, 0) = 0$ ($\ell(z) = \operatorname{Log} z$ is a branch on $\mathbb{C} \setminus \mathbb{R}_-$).



Proof of the Lemma

Let $z(t)$ parametrize γ, $z : [a, b] \to \gamma$. Let

$$h(t) := \int_a^t \frac{z'(s)}{z(s) - z_0}\, ds, \qquad \text{so } h(b) = 2\pi i\, n(\gamma, z_0).$$

Then

$$\left(\frac{e^{h(t)}}{z(t) - z_0}\right)' = e^{h(t)}\, \frac{h'(t)(z(t) - z_0) - z'(t)}{(z(t) - z_0)^2} = e^{h(t)}\, \frac{z'(t) - z'(t)}{(z(t) - z_0)^2} = 0,$$

using $h'(t) = \dfrac{z'(t)}{z(t) - z_0}$. So $\dfrac{e^{h(t)}}{z(t) - z_0}$ is constant:

$$\frac{e^{h(b)}}{z(b) - z_0} = \frac{e^{h(a)}}{z(a) - z_0} = \frac{1}{z(a) - z_0} \qquad (h(a) = 0).$$

Since $z(b) = z(a)$ (closed curve), $e^{h(b)} = 1$, so $h(b) \in 2\pi i\, \mathbb{Z}$. ■